Future 3D Additive Manufacturing
The 3DMM2O Conference

physica status solidi b
www.pss-b.com

Exploring Anomalies by Many-Body Correlations

Klaus Morawetz

The quantum anomaly is written alternatively into a form violating conservation laws or as non-gauge invariant currents seen explicitly on the example of chiral anomaly. By reinterpreting the many-body averaging, the connection to Pauli–Villars regularization is established which gives the anomalous term a new interpretation as arising from quantum fluctuations by many-body correlations at short distances. This is exemplified using an effective many-body quantum potential which realizes quantum Slater sums by classical calculations. It is shown that these quantum potentials avoid the quantum anomaly but approach the same anomalous result by many-body correlations. Consequently, quantum anomalies might be a shortcut way of single-particle field theory to account for many-body effects. This conjecture is also supported since the chiral anomaly can be derived by a completely conserving quantum kinetic theory. A measure for the quality of quantum potentials is suggested to describe these quantum fluctuations in the mean energy. The derived quantum potentials might be used to describe quantum simulations in classical terms.

1. Introduction

1.1. Anomalies

Anomalies are a puzzling discovery in quantum field theory. Certain classical symmetries and conservation laws are broken if the fields become quantized and have been named anomalies. They have a long history starting with investigations of pion decays,[1–3] for an overview see.[4] These anomalies are important for the description of a variety of experiments. In addition to the neutral pion decay also the spontaneously broken axial $U(1)$ symmetry in QCD should be mentioned seen in no parity doubling of baryons and no related Goldstone boson[5] as well as the Kaon decay.[6] The chiral anomaly as a breaking of chiral symmetry[2,7] has recently gained a renewed interest in condensed matter physics as the excitation of chiral mass-less Fermions in the class of Weyl semi-metals.[8–13] It was predicted in ref.[14] and experimentally interpreted[11,15,16] as having observed a chiral anomaly. This has led to an enormous theoretical activity[17–19] describing, for example, anomalous transport,[20–25] the relation of chiral anomaly, and quantized Hall effects[26,27] up to chiral heat effect.[28]

There are two kinds of anomalies. First, consistent anomalies lead to anomalous Ward identities[29,30] not guaranteeing renormalizability of the theory[31,32] and gauge invariance.[6,33] Second, the covariant anomalies do not harm the renormalizability of the theory. Examples are trace anomalies[34] or chiral anomalies. Consistent or covariant anomalies[35–39] accept alternatively violation of gauge invariance or violation of conservation laws. A relation between consistent and covariant anomaly can be achieved by Bardeen–Zumino polynomials.[4,35] Here, we will consider covariant anomalies.

Therefore, it is highly desirable to formulate the theory free of anomalies or re-describe the experimental facts by a consistent theory. In this respect, various anomaly cancellations have been proposed. In electro-weak interaction of the standard model the demand of anomaly free formulation restricts the fermionic content.[4] Nonlocal counter-terms of gauge fields have been used to compensate anomalies.[40] Extending the initial phase space[41] or using higher dimensions, cancellations[42] have been also worked out. Sometimes the similarity of axial non-conservation in chiral and in gravitational anomaly has led to the claim that mixed axial-gravitational anomalies are observed and that it violates Lorentz symmetry.[15,16,43] In ref. [44] it has been shown that a proper subtraction scheme of the infrared divergences shows that corresponding extra terms do not appear. Actually, the Lorentz-invariant chiral kinetic theory can be derived from the quantum kinetic approach[25,45–50] leading to the chiral anomaly by many-body correlations. A hint that the anomalies can be possibly explained by many-body effects is also the relation of 2D conformal anomalies and virial expansions[51] and that the Dirac sea[52,53] can describe the Schwinger anomaly.[54]

Despite the well worked out mathematical appearances of anomalies as triangle graphs in field theory,[2,4,29,34,35,55] the physical origin of anomalies is still a matter of debate. Therefore, it is the motivation of this article to draw attention to a possible alternative scenario. Given the experimental facts, we believe that the anomalies describe real physics. However, it is the question of whether it has to appear as anomalous or whether the same physics can be described by ordinary means. Let us employ an analogy. Pairing in superconductors is conveniently described by anomalous propagators to achieve the Gorkov equations or correspondingly the Beliaev equations for Bose condensates. These propagators violate

K. Morawetz
Department of Engineering Physics
Münster University of Applied Sciences
Stegerwaldstrasse 39, 48565 Steinfurt, Germany
E-mail: morawetz@fh-muenster.de

K. Morawetz
International Institute of Physics-UFRN
Campus Universitário Lagoa nova, 59078-970 Natal, Brazil

The ORCID identification number(s) for the author(s) of this article can be found under https://doi.org/10.1002/pssb.202100316.

DOI: 10.1002/pssb.202100316

the number conservation and are inconsistent in the aforementioned sense. Nevertheless, they lead to correct equations. Recently it has been shown that one can arrive at the same equations by a completely conserving theory of multiple corrected T-matrix[56–58] with equivalent results.[59,60] This illustrates that the anomalous propagators are a theoretical shortcut to the right result through adopting inconsistent propagators.

Analogously we will propose here that the quantum anomaly might be a shortcut way to describe correct physics and investigate the anomaly as a many-body correlation phenomenon. We will illustrate this first in terms of the well-discussed chiral anomaly with the help of Pauli–Villars regularization. This will be shown to be realizable by many-body averaging in the second part of the article where we restrict to the non-relativistic quantum anomaly. This nonrelativistic quantum anomaly is a different physics than the illustrative example of the chiral anomaly of the introduction where the latter one appears only for even dimensions. The non-relativistic anomaly as the main part here is analogous to the non-relativistic trace anomaly of the energy-momentum tensor.[61,62] Their common feature is the regularization by Pauli–Villars or proposed alternatively the many-body averaging. From these non-relativistic considerations, an alternative explanation for the appearance of anomalies is suggested by many-body effects.

1.2. Field Theoretical Approach

Relativistic Fermions with zero mass and consequently linear dispersion have a definite chirality by parallel or anti-parallel spin and motion directions.[63] The left and right-handed projections are realized by Dirac matrices $(1 \mp \gamma_5)/2$ with $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. The chiral or axial transformation

$$\Psi'(x) = e^{i\alpha(x)\gamma_5}\Psi(x) \tag{1}$$

leads to the axial current $J_5 = \overline{\Psi}\gamma^\mu\gamma^5\Psi$ which changes the classical action $S' = S + \int \alpha(x)\nabla_\mu J_5^\mu$. This results into the conservation law

$$\nabla_\mu J_5^\mu = 2im\overline{\Psi}\gamma^5\Psi \to 0, \quad \text{for} \quad m \to 0 \tag{2}$$

for massless Dirac particles. The quantum averaging in contrast

$$\begin{aligned}\langle \partial_\mu J_5^\mu \rangle &= 2im\langle \overline{\Psi}\gamma^5\Psi \rangle \to \frac{e^2}{16\pi^2\hbar^2 c}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \\ &= \frac{e^2}{2\pi^2\hbar^2}\mathbf{E}\cdot\mathbf{B} \quad \text{for} \quad m \to 0\end{aligned} \tag{3}$$

shows a non-vanishing anomalous term obviously due to quantum fluctuations in the average.

The origin is best seen from Pauli–Villars regularization where we subtract from the Dirac Lagrangian for Ψ a massive $(M \to \infty)$ field Φ[4,55,64]

$$\mathcal{L} = i\overline{\Psi}(\partial\!\!\!/ - ieA\!\!\!/)\Psi - i\overline{\Phi}(\partial\!\!\!/ - ieA\!\!\!/)\Phi + M\overline{\Phi}\Phi \tag{4}$$

For the chiral current, we calculate $\mathrm{Tr}\gamma^5 G_{12}$ with $(i\partial\!\!\!/_1 - M + eA\!\!\!/_1)G_{12} = -\delta_{12}$ iteratively by $G = G_0 + G_0 eA\!\!\!/ G$ and $G_0 = -(p\!\!\!/ - M)^{-1}$. Due to the trace, the first non-vanishing terms are of fourth-order

$$\begin{aligned}\langle \partial_\mu J_5^\mu \rangle &= 8\varepsilon^{\kappa\lambda\mu\nu}\int \frac{d^4q d^4r}{(2\pi)^4} e^{irx} e r^\nu A^\mu_{q-r} q^\lambda e A^\kappa_{-q} \int \frac{d^4p}{(2\pi)^4} \frac{M^2}{(p^2-M^2)((p+q)^2-M^2)((r+p)^2-M^2)} \\ &= \frac{-1}{4\pi^2\hbar^2}\varepsilon^{\kappa\lambda\mu\nu}\partial_x^\nu eA^\mu \partial_x^\lambda eA^\kappa = -\frac{e^2}{16\pi^2\hbar^2}\varepsilon^{\kappa\lambda\mu\nu}F^{\nu\mu}F^{\lambda\kappa} = \frac{e^2}{2\pi^2\hbar^2}\mathbf{E}\cdot\mathbf{B}\end{aligned} \tag{5}$$

where one calculates the integral in the $M \to \infty$ limit with $\int d^4p/(p^2+M^2)^3 = \pi^2/2M^2$. This means it comes from divergences up to the fourth adiabatic order (renormalization) which can be expressed by anomalous triangle graphs[17,65,66] known as Adler–Jackiw–Bell anomaly.[2,3,52] Please note the perturbative approach by the expansion of Green's functions. The origin is clearly the behavior at small distances or large momenta. This chiral anomaly can be based on anomalous Ward identities which quantum vector or axial vector fields obey. Only exclusively one of them can be made normal.[67] The rate of chirality (5) can be rewritten explicitly either according to (3) in a non-conservative form

$$\partial_t n_5 + \nabla\cdot\mathbf{j} = \frac{e^2}{2\pi^2\hbar^2}\mathbf{E}\cdot\mathbf{B} \tag{6}$$

or by an anomalous current in a conservative form[68]

$$\partial_t n_5 + \nabla\cdot(\mathbf{j} + \mathbf{j}_{\text{anom}}) = 0 \tag{7}$$

Using the vector and scalar potentials $\mathbf{B} = \nabla\times\mathbf{A}$, $\mathbf{E} = -\dot{\mathbf{A}} - \nabla\phi$, the anomalous current

$$\mathbf{j}_{\text{anom}} = \frac{e^2}{2\pi^2\hbar^2}\left(\frac{1}{2}\dot{\mathbf{A}}\times\mathbf{A} - \phi\nabla\times\mathbf{A}\right) \tag{8}$$

is non-gauge-invariant. So one can choose either to accept a non-conserving rate equation (6) or alternatively a conserving rate equation (7) but with a non-gauge-invariant current (8).

1.3. Many-Body Approach

The same anomalous result (3) can be obtained by many-body effects without anomalous behavior. Heuristically it can be seen easily[69] considering a parallel electric and magnetic field that changes the chirality. The Fermi momentum of the right-handed Fermions increases in the electric field $p_F = eEt$ in opposite direction for left-handed ones. The density of left- and right-handed Fermions is the product of the longitudinal phase-space density, $dN_R/dz = p_F/2\pi\hbar$, and the density of Landau levels in the traverse direction, $d^2N_R/dxdy = eB/2\pi\hbar$, such that the rate of chirality $N_5 = N_R - N_L$ is

$$\frac{dn_5}{dt} = \frac{d^4N_5}{dtd^3x} = 2\frac{\dot{p}_F}{2\pi\hbar}\frac{eB}{2\pi\hbar} = \frac{e^2}{2\pi^2\hbar^2}\mathbf{E}\cdot\mathbf{B} \tag{9}$$

which agrees with (3). We see that completely conventional reasoning by many-body effects leads to the same result as obtained by treatment of the anomaly before. Correspondingly, the quantum kinetic derivation of this result without any non-conserving assumptions can be found.[25]

The **EB** term is also the basis of the experimental interpretation[11,15,16] of having observed chiral anomaly and breaking of conservation laws like a mixed axial-gravitational anomaly. The electrodynamics assuming explicitly a chiral breaking term has been treated in ref. [70]. The well-investigated path from symmetry-violating assumptions to final "non-conservation" form[71] was in a sense misleading if one sees it as a unique signal of violation of conservation laws. One cannot conclude backward from the observed term (3) to a symmetry-breaking field-theoretical assumption since (5) appears also by a conserving theory without the described field-theoretical assumptions.[25]

1.4. Conjecture and Outline

We conclude from the example of the chiral anomaly that the physical origin of the anomalous term is the behavior at small distances and the quantum fluctuations by many-body correlations. Consequently one should be able to get the anomalous results by ordinary many-body treatments. We want here to investigate more in detail how the quantum averaging over the wave function can consistently be performed within a many-body treatment. We will concentrate exclusively on nonrelativistic quantum trace anomaly which is certainly different from the chiral anomaly above but share the common feature as to appear alternatively by many-body effects. We will find a quantum anomaly in the nonrelativistic case for the mean energy. This can be considered as the nonrelativistic trace anomaly of field theory since there the trace of the energy-momentum tensor does not vanish after quantization. The interaction in field theory is realized by exchange fields and results nonrelativistically into potentials sometimes directly used as effective potential method.[72] Since we are dealing with nonrelativistic particles we will observe different short-distance behavior of quantum potentials compared to the Coulomb one. We will show that a proper treatment of such many-body averaging renormalizes the divergence at small distances and no anomaly is present. Though getting rid of anomaly the same physical result appears for the observables as if we had used anomalous terms. Therefore the conjecture is proposed that the anomalies are shortcut ways of single-particle field theory to a many-body effect. In the single-particle treatment they appear as anomalous, in the many-body treatment they appear naturally without anomaly.

The outline of the article is as follows. In Section 2, the nonrelativistic anomalies are re-derived by many-body correlations and the partition function. Conditions are discussed depending on the dimensionality, the power law of potentials, and the perturbation order. This will result in an anomalous energy shift of one-eighth of Rydberg. In Section 3, we show that the use of quantum potentials avoids this anomaly but leads to the same energy shift due to the finite value of the potential at small distances. This finite value is caused by quantum fluctuations which are represented by quantum potentials discussed in binary and ternary order. Section 4 summarizes the results. Appendix A provides integrals occurring in the treatment of Section 2. Appendix B discusses the derivation of quantum potentials on binary and ternary levels separately for Maxwellian and for Fermi particles.

2. Anomaly by Partition Function

First, we observe that the averaging over a many-body statistical operator with kinetic $\widehat{E}$ and potential $\widehat{V}$ energy can be written as inverse Laplace transform

$$\begin{aligned} z &= \mathrm{Tr}\,e^{-\beta(\widehat{E}+\widehat{V}-\mu\widehat{N})} = \mathrm{Tr}\int\limits_{-i\infty+\varepsilon}^{i\infty+\varepsilon}\frac{dM}{2\pi i}e^{\beta(M+\mu\widehat{N})}\frac{1}{\widehat{E}+\widehat{V}+M} \\ &= \mathrm{Tr}\int\limits_{-i\infty+\varepsilon}^{i\infty+\varepsilon}\frac{dM}{2\pi i}e^{\beta(M+\mu\widehat{N})}\left(\frac{1}{\widehat{E}+M}-\frac{1}{\widehat{E}+M}\widehat{V}\frac{1}{\widehat{E}+M}\pm\ \ldots\right) \end{aligned} \tag{10}$$

where we expand with respect to the interaction and assume that it commutes with the number operator. The vanishing statistical averaging corresponds to the high-temperature limit $\beta\to 0$ or alternatively to $M\to\infty$. The latter one takes the role of the large mass of Pauli–Villars regularization. To see this, consider the pure quantum (qm) state expectation in D-dimensions

$$\overset{\mathrm{qm}}{\mathrm{Tr}}\,\widehat{A} = \int\frac{d^Dp}{(2\pi\hbar)^D}\langle p|\widehat{A}|p\rangle = \int\frac{d^Dxd^Dx'd^Dp}{(2\pi\hbar)^D}e^{\frac{-i}{\hbar}\mathbf{px}}\langle x|\widehat{A}|x'\rangle e^{\frac{i}{\hbar}\mathbf{px}'} \tag{11}$$

If the observable $\widehat{A}$ does not contain any explicit $\hbar$, one expects $(2\pi\hbar)^D\mathrm{Tr}\widehat{A}-\lim_{\hbar\to 0}(2\pi\hbar)^D\mathrm{Tr}\widehat{A}=0$. A violation of this zero represents the quantum anomaly[73] we will consider here. Please note the subtle interchange of limiting procedure with the phase-space cell which excludes trivial quantum results vanishing in the classical limit.

To investigate this anomaly further, we combine the many-body mixed state (10) and the quantum-mechanical pure state (11) averaging

$$W^{(1)} = (2\pi\hbar)^Dz - \lim_{\hbar\to 0}(2\pi\hbar)^Dz = \sum_{n=0}^{\infty}(-1)^n\int\frac{dM}{2\pi i}e^{\beta M}W_n \tag{12}$$

with

$$W_n = \sum_{n_1}n_1^ne^{\beta\mu n_1}\int d^Dpd^Dx\left[\frac{1}{n_1\frac{(\mathbf{p}-i\hbar\partial_x)^2}{2m}+M}V(x)\frac{1}{n_1\frac{(\mathbf{p}-i\hbar\partial_x)^2}{2m}+M}\cdot\ \ldots\ -\frac{1}{n_1\frac{(p)^2}{2m}+M}V(x)\frac{1}{n_1\frac{(p)^2}{2m}+M}\cdot\ \ldots\right] \tag{13}$$

where we have used the non-relativistic kinetic energy $\hat{E} = \hat{p}^2/2m$ as an example. The relativistic dispersion works analogously. We have commuted the $\exp(-i\mathbf{px}/\hbar)$ factor in (11) through the kinetic energy factors such that $i\hbar\partial_x$ acts only on the next following potentials. The $\cdot\ldots$ signs indicate n multiplication terms according to V. The sum runs over the number of Fermions $n_1 = 0, 1$ or Bosons $n_1 = 0, 1, 2 \ldots$.

Now, we assume a momentum dependence of the Fourier transformed potential in the form of a power law

$$V(q) = E_0 a_0^{D-\alpha}\hbar^\alpha c_d q^{-\alpha} \tag{14}$$

with a typical energy E_0 and length scale a_0. This would be the Bohr radius $a_0 = a_B$ and $E_0 = Ryd = e^2/4\pi\varepsilon_0 a_B$ for Coulomb potentials in $D = 1,2,3$ dimensions with the numerical factor $c_d = 2^{D-1}\pi$.

The Fourier transformation (13) in dimensionless momentum $\mathbf{k} \to \mathbf{k}\sqrt{n_1/2mM}$ reads then

$$W_n = \frac{(2\pi\hbar)^D c_d^n}{2^{\frac{n}{2}(\alpha-D)} E_0^{s+1}} \sum_{n_1} \frac{e^{\beta\mu n_1}}{n_1^{s+1}} M^s I_n^{(D)}(\alpha) \tag{15}$$

with

$$s = \frac{n}{2}(D - \alpha - 2) - 1 \tag{16}$$

and

$$I_n^{(D)}(\alpha) = \int \frac{d^D k_1}{(2\pi)^D k_1^\alpha} \cdots \frac{d^D k_n}{(2\pi)^D k_n^\alpha} \frac{d^D p}{1 + p^2} \delta(\Sigma_{i=1}^n \mathbf{k}_i) \left[\frac{1}{1 + (\mathbf{p} + \mathbf{k}_1)^2} \cdots \frac{1}{1 + (\mathbf{p} + \mathbf{k}_1 + \ldots + \mathbf{k}_n)^2} - \frac{1}{(1 + p^2)^n}\right] \tag{17}$$

One can understand this integral as a ring-graph of n propagators interacting n times with external potentials as illustrated in **Figure 1**, first considered in.[29]

The expression $W^{(1)}$ describes the anomaly in the normalization of the statistical operator or the completeness. The corresponding term for the mean energy is conveniently expressed as $W^{(H)} = -\partial_\beta W^{(1)}$. If we are interested in the anomaly of the momentum, we have an additional momentum factor that requires $s \to s + 1/2$ and a fore-factor $\sqrt{2}\hbar/a_B$ as well as modified integrals (17) by an additional p factor.

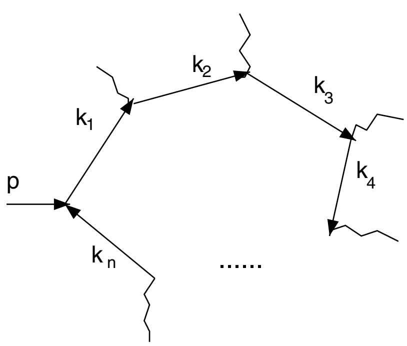

Figure 1. Propagators interacting with external potentials in a ring according to (17).

Performing the inverse Laplace transform (12) we obtain finally

$$W^{(1)} = \sum_{n_1, n=1}^{\infty} \frac{(-c_d)^n (2\pi\hbar)^D}{2^{\frac{n}{2}(\alpha-D)}} \frac{e^{\beta\mu n_1} I_n^{(D)}(\alpha)}{\Gamma(-s)(E_0\beta n_1)^{1+s}} \tag{18}$$

Let us now discuss the $M \to \infty$ or $\beta \to 0$ analogous to the Pauli–Villars regularization. This will produce a non-vanishing anomaly only for certain combinations of dimensions D, power of the momentum in the potential α, and the order of terms n in the sum (18) with the help of (16).

One sees the dependence on β as

$$\begin{aligned} W^{(1)} &= (2\pi\hbar)^D \delta z \sim \beta^{-1-s} \\ W^{(H)} &= (2\pi\hbar)^D \delta H \sim (1+s)\beta^{-2-s} \\ W^{(p)} &= (2\pi\hbar)^D \delta p \sim \beta^{-\frac{3}{2}-s} \end{aligned} \tag{19}$$

which means that one gets with (16) a nonzero anomaly for the normalization δz, the energy δE, and the momentum δp only for the combinations

$$\begin{aligned} &\delta z \neq 0 : \forall n, D = 2 + \alpha \\ &\delta H \neq 0 : n = 1, D = \alpha \text{ or } n = 2, D = \alpha + 1 \\ &\delta p \neq 0 : n = 1, D = 2 + \alpha \end{aligned} \tag{20}$$

For any sensible potential, the relation for the anomaly in the normalization δz is not fulfilled. Also, the case of momentum anomaly is zero due to the vanishing integral (54). We get therefore the anomaly only for the mean value of the energy

$$\langle H \rangle = \frac{Tr\hat{H}\hat{\rho}}{z} + \Delta H, \quad \text{for} \quad n = 1, D = \alpha \quad \text{or} \quad n = 2, D = \alpha + 1 \tag{21}$$

Let us discuss this anomaly in each dimension. In three dimensions $D = 3$ and Coulomb $\alpha = 2$ interaction, the integral (17) is given by (46) and one obtains

$$\Delta H = \frac{W^{(H)}}{(2\pi\hbar)^3} = \frac{1}{8} E_0 \tag{22}$$

a result reported in[73] with a factor of 2 due to spin which we omit here. Summarizing, the quantum anomaly shows up in the mean energy H but not in the normalization z.

In two dimensions with a potential of $\alpha = 1$ one finds analogously

$$W_n^{(1)} \sim \beta^{n/2} \to 0 \quad \text{and} \quad W_n^{(H)} \sim \beta^{n/2-1} \neq 0 \quad \text{for} \quad n = 2 \tag{23}$$

and for $\alpha = 2$

$$W_n^{(1)} \sim \beta^n \to 0 \quad \text{and} \quad W^{(H)} \sim \beta^{n-1} \neq 0 \quad \text{for} \quad n = 1 \tag{24}$$

In one dimension we do not have any anomaly for any potential. Indeed, considering $\alpha = 1$ we have

$$W_n \sim \beta^n \to 0, \quad \text{and} \quad W^{(H)} \sim \beta^{n-1} \neq 0, \quad \text{for} \quad n = 1 \tag{25}$$

but $I_1^{(1)}(1) = 0$ due to (53). For $\alpha = 2$ all anomalies vanish in 1D due to $W_n \sim \beta^{\frac{3}{2}n} \to 0$ and $W_n^{(H)} \sim \beta^{\frac{3}{2}n-1} \to 0$. The Fourier transform of the Coulomb potential in 1D is $\alpha = 0$ but $V(q) \sim \text{sign}(q)$ which leads to $I_2^{(1)}(0) = 0$ as well.

We can conclude that in all discussed cases the anomalies appear due to the large momentum (short distance) divergence of the potential. The results depend on dimensions and the anomaly appears in 2D and 3D but not in 1D and can be understood as a certain limit of the partition function. In the following, we restrict to the discussion of three dimensions.

3. Effective Quantum Potential

We can now state that a finite behavior of the potential at small distances realized by quantum potentials cures such anomalies since we will see that they lead to $\alpha = 4$ which avoids all cases of anomalies (20). These effective quantum potentials appear if we represent the many-body binary quantum correlations by a classical calculation with effective quantum potential. This can be considered at different levels of many-body approximations. Formally there are two ways to achieve this goal. The first way constructs the quantum potential such that the two-particle quantum Slater sum is correctly represented by the classical one with the help of the quantum potential. This results in the Kelbg potential for Maxwellian Coulomb systems[74–77]

$$V_2(r) = \frac{E_0 a_0}{r}\left[1 - e^{-\left(\frac{r}{l}\right)^2} + \sqrt{\pi}\frac{r}{l}\,\text{erfc}\left(\frac{r}{l}\right)\right] = E_0 a_0 \begin{cases} \frac{1}{r} + o(r^{-3}) \\ \frac{\sqrt{\pi}}{l} - \frac{r}{l^2} + o(r^2) \end{cases} \tag{26}$$

where the coordinate is scaled by the thermal wavelength $l^2 = 2\hbar^2/mT$. Improvements and systematic applications can be found in.[78–80]

The second way is to use a statistical equivalence of quantum N-particle systems with an $N+1$-particle classical system.[81] There it was found that the quantum potentials are just successive convolutions of the (Coulomb) potential $V^c(x)$ with the binary distribution $\rho^{(2)}(x)$. If we use the non-degenerate Maxwell correlation

$$\rho^{(2)}(r) = \int \frac{dp}{(2\pi\hbar)^3} e^{ipr/\hbar} e^{-\beta\frac{p^2}{2mT}} = \frac{1}{\pi^{3/2} l^3} e^{-r^2/l^2} \tag{27}$$

the Kelbg potential (26) appears as

$$V_{2ab}(r) \propto \sum_c \int dx_1 \rho_{bc}^{(2)}(x_1) V_{cb}^c(x_1) V_{ca}^c(x_1 + r) \tag{28}$$

with the quantum number, e.q. being charges, indicated by Latin subscripts. These quantum potentials represent quantum (Fock) exchange correlations when employed within the classical energy expression. As calculated in Appendix B, using the Fermi function at $T = 0$, we obtain instead of (27) the potential

$$V_{2f}(r) = \frac{E_0 a_0}{2r}\left[2 + \frac{\pi r}{2 l_F} - \cos\frac{r}{l_F} - \frac{l_F}{r}\sin\frac{r}{l_F} - \frac{r}{l_F} Si\left(\frac{r}{l_F}\right)\right] = E_0 a_0 \begin{cases} \frac{1}{r} + o(r^{-3}) \\ \frac{\pi}{4 l_F} - \frac{r}{6 l_F^2} + o(r^2) \end{cases} \tag{29}$$

where the coordinate scales with the inverse Fermi momentum $l_F = \hbar/p_F$ and the sinus integral is $Si(x) = \int_0^x dt \sin t/t$.

This scheme allows constructing besides the binary quantum potential also the next (ternary) order

$$V_{3ab}(r) \propto \sum_{cd} \int dx_1 dx_2 \rho_{dc}(x_1) V_{cd}^c(x_1) V_{cb}^c(x_1 + x_2) \rho_{bd}(x_2) V_{ca}^c(x_1 + x_2 + r) \tag{30}$$

which leads with (27) to

$$V_3(r) = \frac{E_0 a_0}{r}\left[\text{erf}^2\left(\frac{x}{\sqrt{2}}\right) + \frac{2^{3/2} x}{\sqrt{\pi}} \int_x^\infty \frac{dz}{z} e^{-\frac{z^2}{2}} \text{erf}\left(\frac{z}{\sqrt{2}}\right)\right] = E_0 a_0 \begin{cases} \frac{1}{r} + o(r^0) \\ \frac{\sqrt{8}\ln(1+\sqrt{2})}{\sqrt{\pi} l} - \frac{2r}{\pi l^2} + o(r^2) \end{cases} \tag{31}$$

calculated in Appendix B. The corresponding ternary order for Fermi correlations reads

$$V_{3f}(r) = \frac{E_0 a_0}{r} \frac{4}{\pi^2 + 4} \int_0^1 \frac{dx}{x}\left[x + (1 - x^2)\,\text{artanh}\, x\right] \times \left[2 + \pi x r/l_F - 2\cos(xr/l_F) - 2x\frac{r}{l_F}\text{Si}(xr/l_F)\right] = E_0 a_0 \begin{cases} \frac{1}{r} + o(r^0) \\ \frac{4\pi(1 + 2\ln 2)}{3(4 + \pi^2) l_F} - \frac{2r}{(4 + \pi^2) l_F^2} + o(r^2) \end{cases} \tag{32}$$

The results (29) and (32) are not yet reported while the ones (26) and (31) had been presented in.[81] The quantum potential of binary and ternary correlations are compared in **Figure 2** where the finite limits at small distances are shown. The ternary order somewhat improves the binary quantum potential and leads to somewhat less binding behavior in the attractive case.

This means that the Coulomb divergence at small distances is cured due to quantum fluctuations brought by many-body correlations. Please note that this is the opposite limit than the large-distance Coulomb behavior which is cured by screening. This finite behavior at a small distance translates into a faster potential decay at large momenta. The Fourier transform of the binary potentials read

$$V_2(q) = \frac{8\pi E_0 a_0 \hbar^3}{q^3 l} D\left(\frac{ql}{2\hbar}\right) = E_0 a_0 \begin{cases} \frac{8\pi\hbar^4}{l^2 q^4} + o(q^{-6}) \\ \frac{4\pi\hbar^2}{q^2} - \frac{2\pi l^2}{3} + o(q^2) \end{cases} \tag{33}$$

and

$$V_3(q) = \frac{32E_0 a_0 \hbar^3}{\sqrt{\pi} q^3 l} \int_0^\infty du e^{-u^2} D(u) \ln \left| \frac{(u + \frac{ql}{2})}{(u - \frac{ql}{2})} \right|$$
$$= E_0 a_0 \begin{cases} \frac{16\hbar^4}{l^2 q^4} + o(q^{-6}) \\ \frac{4\pi\hbar^2}{q^2} - \frac{2(2+\pi)}{3} l^2 + o(q^2) \end{cases} \quad (34)$$

with the Dawson function $D(x) = e^{-x^2} \int_0^x dy e^{y^2}$. Analogously one obtains for the Fermi potential

$$V_{2f}(q) = \frac{2\pi E_0 a_0 \hbar^2}{q^2} \left(1 + \frac{p_F^2 - q^2}{2qp_F} \ln \left| \frac{p_F + q}{pf - q} \right| \right)$$
$$= E_0 a_0 \begin{cases} \frac{4\pi\hbar^4}{3 l_F^2 q^4} + o(q^{-5}) \\ \frac{4\pi\hbar^2}{q^2} - \frac{4\pi l_F^2}{3} + o(q^2) \end{cases} \quad (35)$$

and

$$V_{3f}(q) = \frac{16\pi E_0 a_0 \hbar^3}{(4+\pi^2) q^3 l} \int_0^1 dx [x + (1 - x^2) \operatorname{artanh} x] \ \ln \frac{|xp_F + q|}{|xp_F - q|}$$
$$= E_0 a_0 \begin{cases} \frac{16\pi\hbar^4}{(4+\pi^2) l_F^2 q^4} + o(q^{-6}) \\ \frac{4\pi\hbar^2}{q^2} - \frac{4\pi(12+\pi^2)}{3(4+\pi^2)} l_F^2 + o(q^2) \end{cases} \quad (36)$$

These quantum potentials in momentum space are compared in **Figure 3**. The potentials with Fermi correlations show a faster decay around the Fermi momentum $q = p_F$ compared to the Maxwellian ones. We see that all the quantum potentials have a $V(q) \sim 1/q^4$ behavior for large q and a Coulomb behavior at small q. Therefore these quantum potentials lead to $\alpha = 4$ and according to the aforementioned discussions, the anomalies vanish in all cases. Does this mean that the energy shift of the anomaly (22) does not exist? We will see how that will reappear quite ordinarily as the difference of the total energy calculated by quantum potentials compared with the one by the Coulomb potential.

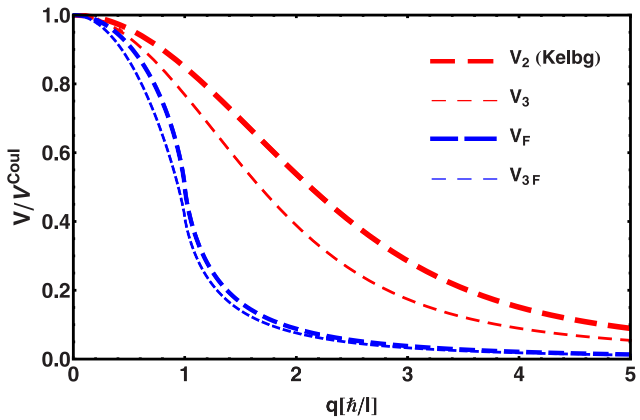

Figure 3. The ratio of the quantum potentials to the Coulomb one in momentum space. For the Kelbg and ternary potential, the scale is $l = \hbar\sqrt{2\beta/m}$ and for the Fermi potential $l = l_F = \hbar/p_F$.

The reason for quantum anomaly is the short distance behavior. Therefore the interaction energy dominates the kinetic energy and we discuss only the correlational energy. Employing the quantum potentials, we could use now any more refined correlation energy but want to restrict the discussion to the lowest-order Hartree correlational energy for homogeneous systems

$$E_{corr} = \frac{1}{2} \int d^3 r' n(r - r') V(r') = \frac{n}{2} V(q = 0) \quad (37)$$

which is the convolution of the potential with the particle density n. According to (33)–(36), the difference of one-particle energies between the quantum potentials and the Coulomb one is

$$\langle \Delta E \rangle = \frac{n}{2} [V_{quant}(q = 0) - V^c(q = 0)]$$
$$= -E_0 \begin{cases} \frac{1}{3\sqrt{\pi}} \frac{a_0}{l} & \text{for} \quad V_2 \\ \frac{2+\pi}{3\pi^{3/2}} \frac{a_0}{l} & \text{for} \quad V_3 \\ \frac{2}{9\pi} \frac{a_0}{l_F} & \text{for} \quad V_{2f} \\ \frac{2(12+\pi^2)}{9\pi(4+\pi^2)} \frac{a_0}{l_F} & \text{for} \quad V_{3f} \end{cases} \quad (38)$$

where we have used the densities $n = 1/\pi^{3/2} l^3$ for $V_{2,3}$ and $n = 1/3\pi^2 l_F^3$ for $V_{2,3f}$.

In contrast, we have the potential energy of a single particle $V(r = 0)$ as indicated in Figure 2

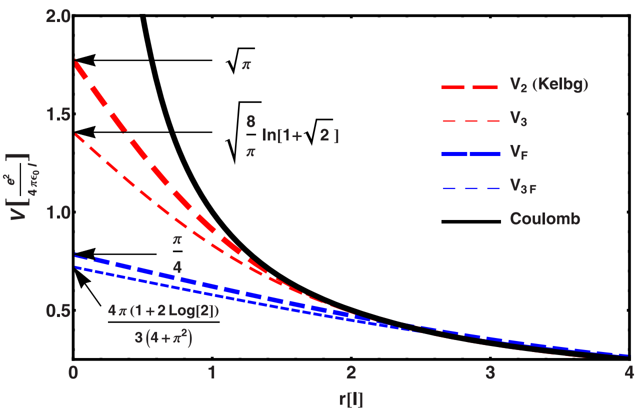

Figure 2. The comparison of the Kelbg potential (26) of binary correlations together with the next (ternary) order correlation potential (31), the Fermi potential (29), and the Coulomb potential. For the Kelbg and ternary potential, the scale is $l = \hbar\sqrt{2\beta/m}$ and for the Fermi potential $l = l_F = \hbar/p_F$. The finite value at zero distance is explicitly indicated.

$$\langle E_1\rangle = V(r=0) = E_0\begin{cases}\sqrt{\pi}\dfrac{a_0}{l} & \text{for} \quad V_2\\ \sqrt{\dfrac{8}{\pi}}\ln(1+\sqrt{2})\dfrac{a_0}{l} & \text{for} \quad V_3\\ \dfrac{\pi}{4}\dfrac{a_0}{l_F} & \text{for} \quad V_{2f}\\ \dfrac{4\pi(1+2\ln 2)}{3(4+\pi^2)}\dfrac{a_0}{l_F} & \text{for} \quad V_{3f}\end{cases} \tag{39}$$

Dividing now (38) by (39), $\frac{\langle\Delta E\rangle}{\langle E_1\rangle} = -\frac{1}{c}$, we get

$$\frac{\Delta H}{\langle E_1^c\rangle} = -\frac{\langle\Delta E\rangle}{\langle E_1^c\rangle} = \frac{1}{1+c} \tag{40}$$

with

$$c = \begin{cases} 3\pi & \approx & 9.24 & \text{for } V_2\\ \dfrac{6\sqrt{2}\ln(1+\sqrt{2})}{(2+\pi)} & \approx & 4.57 & \text{for } V_3\\ \dfrac{9\pi^2}{8} & \approx & 11.10 & \text{for } V_{2f}\\ \dfrac{6\pi^2(1+2\ln 2)}{12+\pi^2} & \approx & 6.46 & \text{for } V_{3f}\end{cases} \tag{41}$$

Comparing with the exact "anomalous" result (22) we would expect $c = 7$. The increasing quality of potentials from binary to ternary level is visible given the lowest-order mean-field we have considered. The Maxwellian is less accurate than the Fermi correlation since we had considered Fermi ones in Section 2. We can now suggest using this ratio c of the energy to the deviation of the energy with quantum potentials from the classical Coulomb one as a measure for the quality of the potential to represent quantum fluctuations.

4. Summary

The nonrelativistic quantum anomaly is investigated for combinations of momentum behavior of potentials, dimensionality, and the order of perturbation. It is found that only for the energy an anomalous shift appears in three dimensions while in one dimension no anomaly is seen. In two dimensions the discussion can be performed analogously. The quantum anomaly appears as the large momentum or short distance behavior of the potential. Quantum potentials are proposed which describe quantum features on a classical level. These quantum potentials lead to a finite value at small distances and cure the Coulomb divergence. The consequence is that no quantum anomaly appears. In contrast, the deviation of the energy with quantum potentials from the energy with the Coulomb potential reflects this anomalous energy shift. In this way, the quantum anomalous behavior is reformulated by normal quantum many-body correlations. It may be a hint that anomalies as such are a theoretical shortcut to the right physics but can be formulated equivalently by a more refined many-body treatment. This of course needs further investigation on a more abstract level than considered here. The discussed quantum potentials might be useful to describe the simulation of strongly correlated quantum systems in classical terms.

Appendix A: Integrals

The occurring integrals in chapter II have the form

$$\begin{aligned} I_n^{(D)}(\alpha) = &\sum_{p,k_1,\ldots k_n}\frac{\delta(k_1+\ldots+k_n)}{k_1^\alpha\ldots k_n^\alpha}\frac{1}{1+p^2}\\ &\left(\frac{1}{1+(\mathbf{p}+\mathbf{k}_1)^2}\cdots\frac{1}{1+(\mathbf{p}+\mathbf{k}_1+\ldots+\mathbf{k}_n)^2} - 1\frac{1}{(1+p^2)^n}\right)\end{aligned} \tag{42}$$

with $\sum_p = \int d^Dp/(2\pi)^D$. Introducing new variables $\mathbf{p}_1 = \mathbf{k}_1 + \mathbf{p}$, $\mathbf{p}_2 = \mathbf{k}_2 + \mathbf{p}_1$ etc. leads to

$$\begin{aligned} I_n^{(D)}(\alpha) = &\sum_{p,p_1\ldots p_{n-1}}\left[\frac{1}{1+p_1^2}\cdots\frac{1}{1+p_{n-1}^2} - \frac{1}{(1+p^2)^{n-1}}\right]\\ &\frac{1}{(\mathbf{p}_1-\mathbf{p})^\alpha}\frac{1}{(\mathbf{p}_2-\mathbf{p}_1)^\alpha}\cdots\frac{1}{(\mathbf{p}_{n-1}-\mathbf{p}_{n-2})^\alpha}\frac{1}{(\mathbf{p}-\mathbf{p}_{n-1})^\alpha}\frac{1}{(1+p^2)^2}\end{aligned} \tag{43}$$

We are going to calculate the integrals for Coulomb potentials $\alpha = 2$.

A.1 3D

Let us consider the integrals with increasing n starting with the lowest non-vanishing one

$$I_2^{(3)}(2) = \sum_{p,p_1}\frac{1}{(\mathbf{p}_1-\mathbf{p})^4}\frac{1}{(1+p^2)^2}\left[\frac{1}{1+p_1^2} - \frac{1}{1+p^2}\right] \tag{44}$$

First performing the integrals about p_1

$$\begin{aligned} &\frac{1}{4\pi^2}\int_0^\infty dp_1 p_1^2\int_{-1}^1\frac{dx}{(p_1^2+p^2-2p_1px)^2}\frac{1}{1+p_1^2}\\ &= \frac{1}{4\pi^2}\int_{-\infty}^\infty dp_1\frac{p_1^2}{(p_1^2-p^2)^2}\frac{l}{1+p_1^2}\end{aligned} \tag{45}$$

where for the second integral in (44) we do not have the last term in (45). Using the residue calculus we circumvent the poles $p_1 = \pm p$ by a semicircle with vanishing radius ε and obtain for (44)

$$\begin{aligned} I_2^{(3)}(2) = &\frac{1}{4\pi^2}\sum_p\left\{\frac{1}{(1+p^2)^2}\left[\frac{1}{\varepsilon}\frac{1}{1+p^2} - \frac{\pi}{(1+p^2)^2}\right]\right.\\ &\left.-\frac{1}{\varepsilon}\frac{1}{(1+p^2)^3}\right\} = -\frac{1}{8\pi^3}\int_0^\infty dp\frac{p^2}{(1+p^2)^4} = -\frac{1}{256\pi^2}\end{aligned} \tag{46}$$

and the divergence cancels leading to a finite result.

All next-order $n > 2$ integrals are divergent. This can be seen from the second part of (43) which is convolution and which can be written as Fourier transform of

$$I_n^{(3)}(2)_{\text{right}} = \int d^3r\left(\frac{1}{4\pi r}\right)^n\sum_p\frac{1}{(1+p^2)^{n+1}} \tag{47}$$

This is obviously divergent at a small distance of the potential due to the powers $n > 2$. The first part of (43) instead is convergent as one can see, e.g. by calculating

$$I_3^{(3)}(2)_{\text{left}} = \sum_{p,p_1,p_2} \frac{1}{(\mathbf{p}_1-\mathbf{p})^2}\frac{1}{(\mathbf{p}_2-\mathbf{p}_1)^2}\frac{1}{(\mathbf{p}-\mathbf{p}_2)^2}\frac{1}{(1+p^2)^2} \times \frac{1}{1+p_1^2}\frac{1}{1+p_2^2} \tag{48}$$

The integration over p_1 can be performed with the help of the Fourier transformation and shifting $\mathbf{k}_1 = \mathbf{p}_1 - \mathbf{p}$

$$\int \frac{d^3p_1}{(2\pi)^3}\frac{1}{(\mathbf{p}_1-\mathbf{p})^2}\frac{1}{(\mathbf{p}_2-\mathbf{p}_1)^2}\frac{1}{1+p_1^2} = \int \frac{d^3r d^3r'}{(4\pi)^2}\int_0^\infty \frac{dk_1}{2\pi^2}\frac{\sin(k_1|r-r'|)}{k_1} \mathrm{e}^{-\mathrm{i}\mathbf{r}\cdot(\mathbf{p}_2-\mathbf{p})-\mathrm{i}\mathbf{r}'\cdot\mathbf{p}-r'} \tag{49}$$

Performing the integration over k_1 which gives $\pi/2$ and after shifting $\mathbf{r} = \mathbf{s} + \frac{1}{2}\mathbf{r}'$ we can use

$$\int_0^\infty ds \frac{s\sin(sa)}{s^2 - \frac{r'^2}{4}} = \frac{\pi}{2}\cos\left(\frac{ar'}{2}\right) \tag{50}$$

to obtain finally

$$\frac{1}{4}\int_0^\infty dr' \frac{\cos\frac{|p-p_2|r'}{2}}{|p-p_2|}\frac{\sin\frac{|p+p_2|r'}{2}}{|p+p_2|}\frac{\mathrm{e}^{-r'}}{r'} = \frac{\pi}{8}\frac{1}{p^2-p_2^2} \tag{51}$$

where we have used

$$\begin{aligned}\int_0^\infty dr' \cos(ar')\sin(br')\frac{\mathrm{e}^{-r'}}{r'} &= \mathrm{Im}\frac{1}{4}\int_{-\infty}^\infty dr' \frac{\mathrm{e}^{-|r'|}}{r'}(\mathrm{e}^{i(a+b)r} + \mathrm{e}^{i(b-a)r}) \\ &= -\frac{1}{4}\mathrm{Im}\, i\int_\pi^0 d\phi(1+1) = \frac{\pi}{2}\end{aligned} \tag{52}$$

A.2 Integrals for $n = 1$

For $n = 1$ the integral (42) takes the form

$$\begin{aligned} I_1^{(D)}(\alpha) &= \sum_{p,k_1}\frac{\delta^D(\mathbf{k}_1)}{k_1^\alpha}\frac{1}{1+p^2}\left(\frac{1}{1+(\mathbf{p}+\mathbf{k}_1)^2} - \frac{1}{(1+p^2)^n}\right) \\ &= -\sum_{p,k_1}\frac{\delta^D(\mathbf{k}_1)}{k_1^\alpha}\frac{k_1^2+2\mathbf{p}\cdot\mathbf{k}_1}{(1+p^2)^3} - = -\sum_{p,k_1}\delta^D(k_1)\frac{k_1^{2-\alpha}}{(1+p^2)^3}\end{aligned} \tag{53}$$

where the integration over p renders the scalar product zero. A non-vanishing result is only for $\alpha = 2$ which means that for one, two, and three dimensions the case $D = \alpha - 2$ is zero

$$I_1^{(D)}(D-2) = 0 \tag{54}$$

and for $D = \alpha$ the only finite result is

$$I_1^{(2)}(2) = -\frac{1}{16\pi^2} \tag{55}$$

due to trivial integrations. In three dimensions there is no potential with $\alpha = 3$.

Appendix B: Quantum Potentials

In the following, we indicate potentials without the units of $E_0 a_0$ by $\overline{V}$. A further possible fore-factor is added in the end corresponding to the demand that the Coulomb potential should be approached for large distances.

B.1 Binary Potentials

We calculate the convolution

$$\overline{V}_2(r) = \int d^3x \rho^{(2)}(x)\overline{V}^c(x)\overline{V}^c(\mathbf{x}+\mathbf{r}) \tag{56}$$

between the binary correlation $\rho^{(2)}$ and the Coulomb potential $\overline{V}^c(r) = 1/r$. The angular integration is trivial

$$\int d\omega \overline{V}^c(\mathbf{x}+\mathbf{r}) = 2\pi\int_{-1}^1 \frac{dz}{\sqrt{x^2+r^2+2xrz}} = 4\pi\begin{cases} x^{-1}, x > r \\ r^{-1}, x < r\end{cases} \tag{57}$$

such that

$$\overline{V}_2(r) = 4\pi\left(\frac{1}{r}\int_0^r dx x\rho^{(2)}(x) + \int_r^\infty dx\rho^{(2)}(x)\right) \tag{58}$$

B.1.1 Maxwellian Correlations

The Maxwellian correlation is the Fourier transform of the Maxwell distribution

$$\rho^{(2)}(r) = \int \frac{d^3p}{(2\pi\hbar)^3}\mathrm{e}^{\frac{i}{\hbar}\mathbf{r}\mathbf{p} - \frac{p^2}{2mT}} = \frac{1}{\pi^{3/2}l^3}\mathrm{e}^{-x^2/l^2} \tag{59}$$

with the thermal wavelength $l^2 = 2\hbar/mT = \lambda^2/\pi$. One easily integrates (58) with the result

$$\begin{aligned}\overline{V}_2(r) &= \frac{2}{\sqrt{\pi}lr}\left[1 - \mathrm{e}^{-\left(\frac{r}{l}\right)^2} + \sqrt{\pi}\frac{r}{l}\mathrm{erfc}\left(\frac{r}{l}\right)\right] \\ &= \frac{2}{\sqrt{\pi}l}\begin{cases}\frac{\sqrt{\pi}}{l} - \frac{r}{l^2} + o(r^2) \\ \frac{l}{r} + o(r^{-5})\end{cases}\end{aligned} \tag{60}$$

which provides the fore-factor $E_0 a_0 \sqrt{\pi} l/2$ to obtain (26). This fore-factor is chosen such that the Coulomb potential appears for large r.

The Fourier transform into momentum space is in principle straightforward. However, we will use the convolution structure since this will turn out to be helpful later for the ternary potentials. The potential (56) translates into a product

$$\overline{V}_2(q) = \{\rho^{(2)}\overline{V}^c\}(q)\overline{V}^c(-q) \tag{61}$$

where we need to calculate the convolution $\{\rho^{(2)}\overline{V}^c\}(q)$. For Maxwellian correlations the angular integration is trivial and one gets

$$\begin{aligned} \int \frac{d^3q'}{(2\pi\hbar)^3}\overline{V}^c(q')\rho^{(2)}(q-q') &= 4\pi\hbar^2 \int \frac{d^3q'}{(2\pi\hbar)^3}\frac{e^{\frac{-l^2(q-q')^2}{4\hbar^2}}}{q'2} \\ &= \frac{2\hbar}{\pi q l^2}\int_0^\infty \frac{dt}{t}\left(e^{-\left(\frac{ql}{2\hbar}-t\right)^2} - e^{-\left(\frac{ql}{2\hbar}+t\right)^2}\right) = \frac{4\hbar}{\sqrt{\pi}l^2 q} D\left(\frac{ql}{2\hbar}\right) \end{aligned} \tag{62}$$

with the Dawson integral $D(x) = e^{-x^2}\int_0^x dy e^{y^2}$. With (61) we obtain

$$\overline{V}_2(q) = \frac{2}{\sqrt{\pi}l}\frac{8\pi\hbar^3}{q^3l^2}D\left(\frac{ql}{2\hbar}\right) = \frac{2}{\sqrt{\pi}l}\frac{4\pi\hbar^2}{q^2}\begin{cases} 1-\frac{l^2}{6\hbar^2}q^2+o(q^3) \\ \frac{2\hbar^2}{l^2q^2}+o(q^{-3}) \end{cases} \tag{63}$$

and we see again the fore-factor $E_0a_0\sqrt{\pi}l/2$ to get the expression (33) including the expansions.

B.1.2 Fermi Correlations

With Fermi correlations as Fourier transform of the Fermi function

$$\rho_{2f}(x) = \int_{p\le p_F}\frac{d^3p}{(2\pi\hbar)^3}e^{\frac{i}{\hbar}rp} = \frac{1}{2\pi^2r^3}\left(\sin\frac{r}{l_F} - \frac{r}{l_F}\cos\frac{r}{l_F}\right) \tag{64}$$

with $l_F = \hbar/p_F$, one obtains for (58)

$$\begin{aligned} V_{2f}(r) &= \frac{1}{\pi r l_F}\left\{2-\cos\overline{r} - \frac{\sin\overline{r}}{\overline{r}} + \overline{r}\left[\frac{\pi}{2} - Si(\overline{r})\right]\right\} \\ &= \begin{cases} \frac{1}{2l_F^2} - \frac{r}{3\pi l_F^3} + o(r^2) \\ \frac{2}{\pi r l_F} + o(r^{-3}) \end{cases} \end{aligned} \tag{65}$$

with $\overline{r} = r/l_F$. We see that the fore-factor to be chosen is here $E_0a_0\pi l_F/2$ to obtain (29).

The Fourier transform we calculate analogously to the Maxwellian with

$$\begin{aligned} \{\rho_{2f}\overline{V}^c\}(p) &= \int\frac{d^3q}{(2\pi\hbar)^3}\overline{V}^c(p-q)\rho_{2f}(q) = \frac{1}{2\pi\hbar p}\int_0^{p_F} dq q \ln\frac{(p+q)^2}{(p-q)^2} \\ &= \frac{p_F}{2\pi\hbar}\left[2+\left(\frac{p_F}{p}-\frac{p}{p_F}\right)\ln\frac{|p+p_F|}{|p-p_F|}\right] \end{aligned} \tag{66}$$

where we have used the trivial angular integration

$$\int d\Omega\overline{V}^c(p-q) = \int_{-1}^{1} dx\frac{8\pi^2\hbar^2}{q^2+p^2-2pqx} = \frac{4\pi^2\hbar^2}{pq}\ln\frac{(p+q)^2}{(p-q)^2} \tag{67}$$

With the help of (66) we obtain for (61)

$$\begin{aligned} \overline{V}_{2f}(q) &= \frac{2\hbar^2}{l_Fq^2}\left[2+\left(\frac{p_F}{p}-\frac{p}{p_F}\right)\ln\frac{|p+p_F|}{|p-p_F|}\right] \\ &= \frac{8\hbar^2}{l_Fq^2}\begin{cases} 1-\frac{q^2}{3p_F^2}+o(q^4) \\ \frac{p_F^2}{3q^2}+o(q^{-4}) \end{cases} \end{aligned} \tag{68}$$

which provides again the fore-factor $E_0a_0\pi l_F/2$ as above to get finally (35).

B.2 Ternary Potentials

The convolution structure of the ternary potentials (30) suggests calculating them in momentum space

$$\overline{V}_3(q) = V(-q)\int\frac{d^3p}{(2\pi\hbar)^3}\{\rho^{(2)}\overline{V}^c\}(p)\rho^{(2)}(p)\overline{V}^c(q-p) \tag{69}$$

where we can conveniently use the results of the foregoing chapter (62) or (66), respectively.

B.2.1 Maxwellian Correlations

Introducing the simple angular integration (67) into (69) and using (62) we obtain

$$\begin{aligned} \overline{V}_3(q) &= \frac{2}{\sqrt{\pi}}\int_0^\infty du e^{-u^2}D(u)\ln\frac{(u+\frac{ql}{2\hbar})^2}{(u-\frac{ql}{2\hbar})^2} \\ &= \frac{4\pi\hbar^2}{l^2q^2}\begin{cases} 1-\frac{(2+\pi)q^2l^2}{6\pi\hbar^2}+o(q^4) \\ \frac{4\hbar^4}{\pi q^2l^4}+o(q^{-4}) \end{cases} \end{aligned} \tag{70}$$

where we used the integrals

$$\int_0^\infty e^{-u^2}D(u)\begin{pmatrix} u \\ u^{-1} \\ u^{-3} \end{pmatrix} = \frac{\sqrt{\pi}}{8}\begin{pmatrix} 1 \\ \pi \\ -2(2+\pi) \end{pmatrix} \tag{71}$$

for the small and large-q expansions. The comparison with the Coulomb potential for small q provides the fore-factor $E_0a_0l^2$ to get just (34).

For the potential in the spatial domain, we integrate directly (30) with a trivial renaming $\mathbf{x}_1 \to \mathbf{x}, \mathbf{x}_2 \to \mathbf{y}-\mathbf{x}$

$$V_3(r) = \int d^3xd^3y\rho^{(2)}(x)V^c(x)V^c(y)\rho^{(2)}(\mathbf{y}-\mathbf{x})V^c(\mathbf{y}+\mathbf{r}) \tag{72}$$

The angular integration of y is given by (57) and the one of x by

$$\int d\Omega_x\rho^{(2)}(\mathbf{y}-\mathbf{x}) = \frac{2\sinh\frac{2xy}{l^2}}{\pi^2l^4xy}e^{\frac{-(x^2+y^2)}{l^2}} \tag{73}$$

The $|x|$ integration yields error functions. Using $y = lt$ the remaining integration reads

$$V_3(r) = \frac{2^{3/2}}{\sqrt{\pi} l^3}\left(\int_{r/l}^{\infty}\frac{dt}{t} + \frac{1}{r}\int_0^{r/l} dt\right) e^{-\frac{t^2}{2}}\mathrm{erf}\left(\frac{t}{\sqrt{2}}\right) \tag{74}$$

The second integral can be made analytically observing that one can perform a partial integration $\bar{r} = r/l$

$$\begin{aligned}\int_0^{\bar r} dt\left[\sqrt{\frac{\pi}{2}}\mathrm{erf}\left(\frac{t}{\sqrt{2}}\right)\right]\mathrm{erf}\left(\frac{t}{\sqrt{2}}\right) &= \sqrt{\frac{\pi}{2}}\mathrm{erf}^2\left(\frac{t}{\sqrt{2}}\right)\Big|_0^{\bar r}\\ -\sqrt{\frac{\pi}{2}}\int_0^{\bar r} dt\,\mathrm{erf}\left(\frac{t}{\sqrt{2}}\right)\sqrt{\frac{2}{\pi}}e^{-\frac{t^2}{2}} &= \sqrt{\frac{\pi}{8}}\mathrm{erf}^2\left(\frac{\bar r}{\sqrt{2}}\right)\end{aligned} \tag{75}$$

The last identity appears just observing that the second term from the partial integration is just the negative of the desired integral itself. Finally, we obtain

$$\begin{aligned}\overline{V}_3(r) &= \frac{1}{l^3}\left[2^{3/2}\frac{r}{\sqrt{\pi} l}\int_{r/l}^{\infty}\frac{dt}{t}e^{-\frac{t^2}{2}}\mathrm{erf}\left(\frac{t}{\sqrt{2}}\right) + \mathrm{erf}^2\left(\frac{r}{\sqrt{2} l}\right)\right]\\ &= \frac{1}{l^2}\begin{cases}\frac{\sqrt{8}\ln(1+\sqrt{2})}{\sqrt{\pi} l} - \frac{2}{\pi}\frac{r}{l^2} + \frac{2r^2}{3\pi l^3} + o(r^3)\\ \frac{l}{r} + o(r^{-2})\end{cases}\end{aligned} \tag{76}$$

This shows again the fore-factor $E_0 a_0 l^2$ to get (31).

B.2.2 Fermi Correlations

For Fermi correlations, we introduce (66) and (67) into (69) to obtain

$$\begin{aligned}\overline{V}_{3f}(q) &= \frac{4p_F^3}{\pi q^3}\int_0^1 dx[x + (1-x^2)\mathrm{artanh}(x)]\ \ln\frac{|xp_F + q|}{|xp_F - q|}\\ &= \frac{p_F^2(4+\pi^2)}{4\pi^2\hbar^2}\begin{cases}\frac{16\hbar^4}{(4+\pi^2)l_F^2 q^4} + o(q^{-6})\\ \frac{4\pi\hbar^2}{q^2} - \frac{4\pi\hbar^2}{3p_F^2}\frac{(12+\pi^2)}{(4+\pi^2)} + o(q^2)\end{cases}\end{aligned} \tag{77}$$

Choosing the fore-factor as $E_0 e_0 4\pi^2\hbar^2/p_F^2(4+\pi^2)$ such that for small momentum q the Coulomb result appears corresponding to large distance behavior, we obtain just (36).

The first term of the small q-expansion of (77) and large q-expansion can be performed directly using

$$\int_0^1 dx[x + (1-x^2)\mathrm{artanh}(x)]\begin{pmatrix}x\\1\\x^{-1}\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ \frac{1}{3}(1+2\ln 2)\\ \frac{1}{8}(4+\pi^2)\end{pmatrix} \tag{78}$$

The second term in the small-q expansion of (36) deserves special attention. The needed integral with x^{-3} in (78) would diverge. The reason is a tricky order of principal value integrations. The best way to solve this problem is to consider a Debye potential $\exp(-\kappa r/\hbar)/r$ with a vanishing κ in the angular integration (67) instead of the Coulomb potential $\sim 1/r$

$$\begin{aligned}\int d\Omega\overline{V}^c(\mathbf{q}-\mathbf{p}) &= \int_{-1}^1 dx\frac{8\pi^2\hbar^2}{q^2+p^2-2pqx+\kappa^2}\\ &= \frac{4\pi^2\hbar^2}{pq}\ln\frac{(p+q)^2+\kappa^2}{(p-q)^2+\kappa^2}\end{aligned} \tag{79}$$

This leads instead of (66) to

$$\begin{aligned}\{\rho_{2f}\overline{V}^c\}(p) &= \frac{p_F}{2\pi\hbar}\left[2 - \frac{2e}{x}\arctan\frac{2e}{e^2+x^2-1}\right.\\ &\left.+\left(\frac{e^2-x^2+1}{2x}\right)\ln\frac{(1+x)^2+e^2}{(1-x)^2+e^2}\right]\end{aligned} \tag{80}$$

with $x = p/p_F$ and $e = \kappa/p_F$. Though the limit $e \to 0$ gives (66) the x-integral in (77) diverges if performed after this limit and is finite when the limit is performed after the integration. To see this, we consider the large-q expansion

$$\frac{1}{q^3}\ln\frac{|xp_F+q|}{|xp_F-q|} = \frac{2}{xq^2p_F} + \frac{2}{3x^3p_F^3} + \ldots \tag{81}$$

and have with (80) instead of (77)

$$\begin{aligned}\overline{V}_{3f}(q) &= \frac{8}{\pi}\int_0^1 dx\left[x - e\arctan\frac{2e}{e^2+x^2-1}\right.\\ &\left.+\left(\frac{e^2-x^2+1}{4}\right)\ln\frac{(1+x)^2+e^2}{(1-x)^2+e^2}\right]\left(\frac{p_F^2}{xq^2} + \frac{1}{3x^3} + o(q^2)\right)\end{aligned} \tag{82}$$

The first term $\sim 1/x$ is convergent in the $e \to 0$ limit according to (78). For the second problematic q^0 term $\sim 1/x^3$ we first integrate and then perform the limit with the result

$$\overline{V}_{3f}(q) = \frac{p_F^2}{q^2}\left(\frac{4}{\pi} + \pi\right) - \left(\frac{4}{\pi} + \frac{\pi}{3}\right) + \frac{8}{3\pi}e + o(e^2, q^2) \tag{83}$$

which after $e \to 0$ gives the expansion (77) and (36).

The form (77) is convenient for the Fourier transform which yields

$$\begin{aligned}\overline{V}_{3f}(r) &= \int\frac{d^3q}{(2\pi\hbar)^3}\overline{V}_{3f}(q) = \frac{p_F^2}{\pi^3\hbar^2 r}\int_0^{\infty} dy\frac{\sin y\bar r}{y^2}\int_0^1 dx\\ &[x + (1-x^2)\mathrm{artanh}(x)]\ \ln\frac{(x+y)^2}{(x-y)^2}\end{aligned} \tag{84}$$

with $y = q/p_F$ and $\bar r = rp_F/\hbar$. The y-integration can be performed

$$\int_0^\infty dy \frac{\sin y\bar{r}}{y^2} \ln\frac{(x+y)^2}{(x-y)^2} = \frac{\pi}{x}[2+\pi x\bar{r} - 2\cos x\bar{r} - 2x\bar{r}\mathrm{Si}(x\bar{r})] = \begin{cases} \pi^2\bar{r} - \pi x\bar{r}^2 + o(\bar{r}^3) \\ \frac{2\pi}{x} + o(\bar{r}^{-1}) \end{cases} \quad (85)$$

with the sinus integral $\mathrm{Si}(x) = \int_0^x dt \sin t/t$. Using (78) the needed fore-factor can be seen from the $r \to 0$ expansion

$$\overline{V}_{3f}(r) = \frac{2p_F^2}{\pi^2\hbar^2 r}\int_0^1 dx[x + (1-x^2)\mathrm{artanh}(x)][x^{-1} + o(r)] = \frac{p_F^2(4+\pi^2)}{4\pi^2\hbar^2}\begin{cases} \frac{4\pi(1+2\ln 2)}{3(4+\pi^2)l_F} - \frac{2r}{(4+\pi^2)l_F^2} + o(r^3) \\ \frac{1}{r} + o(r^{-2}) \end{cases} \quad (86)$$

again to be $E_0 e_0 4\pi^2\hbar^2/p_F^2(4+\pi^2)$ to obtain the Coulomb potential for large distances which all together provides (32).

Acknowledgements

Open access funding enabled and organized by Projekt DEAL.

Conflict of Interest

The author declares no conflict of interest.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Keywords

many-body correlations, quantum anomaly, quantum potentials

Received: July 1, 2021
Revised: September 2, 2021
Published online: October 29, 2021

[1] J. Steinberger, *Phys. Rev.* **1949**, *76*, 1180.
[2] S. L. Adler, *Phys. Rev.* **1969**, *177*, 2426.
[3] J. Bell, R. Jackiw, *Nuovo Cimento A* **1969**, *60*, 47.
[4] R. A. Bertlmann, *Anomalies in Quantum Field Theory*, Clarendon Press, Oxford **2000**.
[5] G. 't Hooft, *Phys. Rev. Lett.* **1976**, *37*, 8.
[6] J. Wess, B. Zumino, *Phys. Lett. B* **1971**, *37*, 95.
[7] H. Leutwyler, *Ann. Phys. (N.Y.)* **1994**, *235*, 165.
[8] S.-Y. Xu, I. Belopolski, N. Alidoust, M. Neupane, G. Bian, C. Zhang, R. Sankar, G. Chang, Z. Yuan, C.-C. Lee, S.-M. Huang, H. Zheng, J. Ma, D. S. Sanchez, B. Wang, A. Bansil, F. Chou, P. P. Shibayev, H. Lin, S. Jia, M. Z. Hasan, *Science* **2015**, *349*, 613.
[9] L. Lu, Z. Wang, D. Ye, L. Ran, L. Fu, J. D. Joannopoulos, M. Soljačić, *Science* **2015**, *349*, 622.
[10] B. Q. Lv, H. M. Weng, B. B. Fu, X. P. Wang, H. Miao, J. Ma, P. Richard, X. C. Huang, L. X. Zhao, G. F. Chen, Z. Fang, X. Dai, T. Qian, H. Ding, *Phys. Rev. X* **2015**, *5*, 031013.
[11] S.-M. Huang, S.-Y. Xu, I. Belopolski, C.-C. Lee, G. Chang, B.-K. Wang, N. Alidoust, G. Bian, M. Neupane, C. Zhang, S. Jia, A. Bansil, H. Lin, M. Z. Hasan, *Nat. Commun.* **2015**, *6*, 7373.
[12] H. Weng, C. Fang, Z. Fang, B. A. Bernevig, X. Dai, *Phys. Rev. X* **2015**, *5*, 011029.
[13] Y.-S. Jho, J.-H. Han, K.-S. Kim, *Phys. Rev. B* **2017**, *95*, 205113.
[14] X. Wan, A. M. Turner, A. Vishwanath, S. Y. Savrasov, *Phys. Rev. B* **2011**, *83*, 205101.
[15] S.-Y. Xu, N. Alidoust, G. Chang, H. Lu, B. Singh, I. Belopolski, D. S. Sanchez, X. Zhang, G. Bian, H. Zheng, M.-A. Husanu, Y. Bian, S.-M. Huang, C.-H. Hsu, T.-R. Chang, H.-T. Jeng, A. Bansil, T. Neupert, V. N. Strocov, H. Lin, S. Jia, M. Z. Hasan, *Sci. Adv.* **2017**, *3*, 6.
[16] J. Gooth, A. C. Niemann, T. Meng, A. G. Grushin, K. Landsteiner, B. Gotsmann, F. Menges, M. Schmidt, C. Shekhar, V. Süß, R. Hühne, B. Rellinghaus, C. Felser, B. Yan, K. Nielsch, *Nature* **2017**, *547*, 324.
[17] G. Başar, D. E. Kharzeev, H.-U. Yee, *Phys. Rev. B* **2014**, *89*, 035142.
[18] A. Johansson, J. Henk, I. Mertig, *Phys. Rev. B* **2019**, *99*, 075114.
[19] J. Fröhlich, B. Pedrini, in *Mathematical Physics 2000*, Imperial College Press, London **2000**, pp. 9–47.
[20] A. Y. Alekseev, V. V. Cheianov, J. Fröhlich, *Phys. Rev. Lett.* **1998**, *81*, 3503.
[21] H.-J. Kim, K.-S. Kim, J.-F. Wang, M. Sasaki, N. Satoh, A. Ohnishi, M. Kitaura, M. Yang, L. Li, *Phys. Rev. Lett.* **2013**, *111*, 246603.
[22] K. Landsteiner, *Acta Phys. Pol. B* **2016**, *47*, 2617.
[23] E. V. Gorbar, V. A. Miransky, I. A. Shovkovy, P. O. Sukhachov, *Low Temp. Phys.* **2018**, *44*, 487.
[24] K. Morawetz, *Phys. Rev. B* **2016**, *94*, 165415.
[25] K. Morawetz, *Eur. Phys. J. B* **2019**, *92*, 176.
[26] A. Abouelsaood, *Phys. Rev. Lett.* **1985**, *54*, 1973.
[27] S. Moroz, C. Hoyos, L. Radzihovsky, *Phys. Rev. B* **2015**, *91*, 195409.
[28] T. Kimura, T. Nishioka, *Prog. Theor. Phys.* **2012**, *127*, 1009.
[29] W. A. Bardeen, *Phys. Rev.* **1969**, *184*, 1848.
[30] L. Dolan, R. Jackiw, *Phys. Rev. D* **1974**, *9*, 3320.
[31] D. J. Gross, R. Jackiw, *Phys. Rev. D* **1972**, *6*, 477.
[32] C. Becchi, A. Rouet, R. Stora, in *Field Theory, Quantization and Statistical Physics* (Ed: E. Tirapegui), Springer, Dordrecht **1981**, pp. 3–32.
[33] J. Preskill, *Ann. Phys.* **1991**, *210*, 323.
[34] F. Bastianelli, P. van Nieuwenhuizen, *Nucl. Phys. B* **1993**, *389*, 53.
[35] W. A. Bardeen, B. Zumino, *Nucl. Phys. B* **1984**, *244*, 421.
[36] S. Randjbar-Daemi, J. Strathdee, *Phys. Lett. B* **1997**, *402*, 134.
[37] C. Adam, *Ann. Phys. (N.Y.)* **1998**, *265*, 198.
[38] C. Ekstrand, *Phys. Lett. B* **2000**, *485*, 193.
[39] M. Takeuchi, R. Endo, *Prog. Theor. Exp. Phys.* **2017**, *2017*, 033B03.
[40] N. Krasnikov, *Nuov Cim A* **1985**, *89*, 308.
[41] M. Moshe, Y. Oz, *Phys. Lett. B* **1989**, *224*, 145.
[42] M. B. Green, J. H. Schwarz, P. West, *Nucl. Phys. B* **1985**, *254*, 327.
[43] A. A. Zyuzin, A. A. Burkov, *Phys. Rev. B* **2012**, *86*, 115133.
[44] O. Del Cima, D. Franco, O. Piguet, *J. High Energy Phys.* **2017**, *2017*, 11.
[45] J.-H. Gao, Z.-T. Liang, S. Pu, Q. Wang, X.-N. Wang, *Phys. Rev. Lett.* **2012**, *109*, 232301.
[46] J.-W. Chen, S. Pu, Q. Wang, X.-N. Wang, *Phys. Rev. Lett.* **2013**, *110*, 262301.
[47] C. Manuel, J. M. Torres-Rincon, *Phys. Rev. D* **2014**, *90*, 076007.
[48] J.-h. Gao, S. Pu, Q. Wang, *Phys. Rev. D* **2017**, *96*, 016002.
[49] Y. Hidaka, S. Pu, D.-L. Yang, *Phys. Rev. D* **2017**, *95*, 091901(R).
[50] A. Huang, S. Shi, Y. Jiang, J. Liao, P. Zhuang, *Phys. Rev. D* **2018**, *98*, 036010.
[51] W. Daza, J. E. Drut, C. Lin, C. Ordóñez, *Phys. Rev. A* **2018**, *97*, 033630.
[52] H. Nielsen, M. Ninomiya, *Phys. Lett. B* **1983**, *130*, 389.

[53] C. Adam, R. A. Bertlmann, P. Hofer, *Riv. Nuovo Cim.* **1993**, *16*, 1.
[54] A. A. Vladimirov, *J. Phys. A: Math. Gen.* **1990**, *23*, 87.
[55] K. Fujikawa, H. Suzuki, *Path Integrals and Quantum Anomalies*, Oxford University Press, Oxford **2004**.
[56] P. Lipavský, *Phys. Rev. B* **2008**, *78*, 214506.
[57] K. Morawetz, *J. Stat. Phys.* **2011**, *143*, 482.
[58] B. Šopk, P. Lipavský, M. Männel, K. Morawetz, P. Matlock, *Phys. Rev. B* **2011**, *84*, 094529.
[59] K. Morawetz, *Phys. Rev. B* **2010**, *82*, 092501.
[60] M. Männel, K. Morawetz, P. Lipavský, *Phys. Rev. A* **2013**, *87*, 053617.
[61] O. Bergman, *Phys. Rev. D* **1992**, *46*, 5474.
[62] T. Haugset, F. Ravndal, *Phys. Rev. D* **1994**, *49*, 4299.
[63] H. Weyl, *Z. Phys.* **1929**, *56*, 330.
[64] U. N. Kafiev, *Anomali i teoria strun*, Nauka, Nowosibirsk **1991** [in Russian].
[65] K. Jensen, *Phys. Rev. D* **2012**, *85*, 125017.
[66] K. Landsteiner, *Acta Phys. Pol. B* **2016**, *47*, 2617.
[67] W. A. Bardeen, *Phys. Rev.* **1969**, *184*, 1848.
[68] K. Morawetz, *Phys. Lett. A* **2019**, *383*, 1362.
[69] K. Fukushima, D. E. Kharzeev, H. J. Warringa, *Phys. Rev. D* **2008**, *78*, 074033.
[70] Z. Qiu, G. Cao, X.-G. Huang, *Phys. Rev. D* **2017**, *95*, 036002.
[71] A. A. Zyuzin, S. Wu, A. A. Burkov, *Phys. Rev. B* **2012**, *85*, 165110.
[72] C. L. Lin, C. R. Ordóñez, *Phys. Rev. D* **2016**, *94*, 085001.
[73] D. A. Kirzhnitz, G. V. Shpatakovskaya, *Theor. Math. Phys.* **1996**, *109*, 1342.
[74] G. Kelbg, *Ann. Phys. (Leipzig)* **1964**, *13*, 354.
[75] G. Kelbg, *Ann. Phys. (Leipzig)* **1964**, *14*, 394.
[76] W. Ebeling, H. Hoffmann, G. Kelbg, *Beitr. Plasmaphys.* **1967**, *7*, 233.
[77] D. Kremp, W. D. Kraeft, *Ann. Phys. (Leipzig)* **1968**, *20*, 340.
[78] W. D. Kraeft, D. Kremp, W. Ebeling, G. Röpke, *Quantum Statistics of Charged Particle Systems*, Akademie Verlag, Berlin **1986**.
[79] W. D. Kraeft, D. Kremp, *Z. Phys.* **1968**, *208*, 475.
[80] J. Ortner, I. Valuev, W. Ebeling, *Contrib. Plasma Phys.* **2000**, *40*, 555.
[81] K. Morawetz, *Phys. Rev. E* **2002**, *66*, 022103.